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Resolution of Downstream Boundary Layers in the Chebyshev Approximation to Viscous Flow Problems

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Steady solutions characterized by downstream boundary layers are sought to the one-dimensional linearized Burgers equation for a variety of computational—that is, spatial and temporal differencing—schemes. When well resolved, the presence of these boundary layers does not seriously affect the accuracy of the interior solutions. For narrow outflow boundary layers, however, the Chebyshev collocation method may be unstable, unlike the finite-difference technique, for grid Reynolds numbers less than a certain critical value. Although the spectral solutions are improved by using fractional step time-differencing methods in which the viscous and advective effects are separately treated, the stability of the resulting multistep technique need not be guaranteed by the stability of its component steps. Analogous computational restrictions on the use of the Chebyshev collocation method are shown to hold for the nonlinear Burgers equation.

1. INTRODUCTION

Increasing attention in the fields of computational and geophysical fluid dynamics has been paid to the problem of integrating the hydrodynamic equations over a limited (open) domain [1]. Of persistent concern has been the selection and implementation of appropriate boundary conditions for regions of outflow [2, 3]. Fortunately, Fix and Gunzberger [4] have shown for viscous transport problems that incorrect specification of the outflow boundary condition will adversely affect the solution only in a narrow region adjacent to outflow. This is necessarily true, however, only if sufficient resolution is maintained in the outflow boundary layer. Since it may often be computationally inefficient to fully resolve the outflow region (if, for example, the important dynamical scales are much larger than the boundary layer thickness), it is of interest to inquire as to the effects of insufficient boundary layer resolution.

In this paper, we explore the extent to which accurate, stable, and convergent results can be obtained to a linearized one-dimensional model problem for which a narrow viscous boundary layer exists on the downstream boundary. The prototype problem is solved using both finite-difference and spectral (Chebyshev) approximations

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in space, and one of several stepping schemes in time. The intent is *not* to provide a definitive comparison between the finite-difference and spectral methods when applied to equations of this type, but to investigate the quite different computational restrictions associated with each technique.

The resulting spatial approximations are convergent. As anticipated, however, an accurate and/or stable interior solution can be maintained only if the narrow viscous boundary layer is sufficiently well resolved, that is, if the grid Reynolds number (to be defined) is sufficiently small. Of particular interest is the fact that the Chebyshev approximation is susceptible, unlike the finite-difference schemes studies, to a temporal instability which is eliminated by taking a larger, rather than a smaller, time step. For fixed spatial resolution, the results imply a trade-off between stability and accuracy. As we will see, similar remarks hold for the spectral approximation to fully nonlinear problems.

2. A Linearized Transport Equation

Consider the following advective–diffusive ("linearized Burgers") equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad (0 \leqslant x \leqslant 1). \tag{1}$$

For the purposes of this discussion, we restrict our attention to two problems corresponding to the boundary conditions,

$$u(0, t) = 1, \qquad u(1, t) = 0, \qquad \text{Problem 1 (P1)} \tag{2a}$$

and

$$u(0, t) = 1, \qquad \frac{\partial}{\partial x} u(1, t) = 0, \qquad \text{Problem 2 (P2).} \tag{2b}$$

Problem 1, defined by Eqs. (1) and (2a), has a steady-state solution characterized by a viscous boundary layer of thickness ν near $x = 1$; that is,

$$u(x, t) \xrightarrow[t\to\infty]{} \frac{1 - \exp[(x-1)/\nu]}{1 - \exp(-1/\nu)}. \tag{3a}$$

The second problem has no such steady boundary layer; for (1) and (2b),

$$u(x, t) \xrightarrow[t\to\infty]{} 1. \tag{3b}$$

In the succeeding sections, Eq. (1) will be numerically integrated in time with the initial conditions $u(0, 0) = 1$ and $u(x > 0, 0) = 0$ in search of the steady-state solutions (3a) and (3b). It will be shown that accurate interior solutions can be obtained only if sufficient spatial accuracy is maintained in the outflow boundary layer region and that this conclusion is not changed for problems, such as P2, which have no explicit steady-state boundary layer character.

3. Approximation by the Chebyshev Collocation Method

We will solve P1 and P2 by the Chebyshev collocation method (also called pseudo-spectral approximation). The details of the technique have been given elsewhere [5–7]. We note only the form of the discrete spectral representation,

$$u(x, n\,\Delta t) = u^n(x) = \sum_{p=0}^{N} a_p{}^n T_p(\hat{x}), \qquad \hat{x} = 2x - 1. \tag{4}$$

$T_p(\hat{x}) = T_p(\cos\theta) = \cos(p\theta)$ is the Chebyshev polynomial of degree p. For the present application, the spectral cutoff, N, is 64. As expansion functions, series of Chebyshev polynomials are extremely convenient for two reasons. First, Chebyshev transforms can be interpreted as cosine transforms on the nonuniformly spaced collocation grid $\hat{x}_q = \cos(\pi q/N)$, $0 \leqslant q \leqslant N$. Together with certain well-known Chebyshev recursion relations, this means that the fields u_x and u_{xx} can be evaluated very efficiently, in $O(N \log N)$ operations. Second, the resolution of the collocation grid is enhanced near $x = 0$ and 1, where the narrow viscous layers are expected in these examples. When using explicit time differencing, however, the Courant condition for the maximal allowable time step can be very restrictive given the much finer grid spacing at the edges of the computational domain.

The correct implementation of boundary conditions (2a) or (2b) requires that the diffusive term $\nu(\partial^2 u/\partial x^2) = \nu u_{xx}$ (at least) be treated implicitly. To avoid the necessity of taking very small time steps, it is also advantageous to adopt an implicit representation of u_x. We consider, therefore, the following semi-implicit time-differencing schemes:

(i) Crank–Nicolson (CN)

$$\frac{u^{n+1} - u^n}{\Delta t} = \tfrac{1}{2}[\nu u_{xx} - u_x]^{n+1} + \tfrac{1}{2}[\nu u_{xx} - u_x]^n, \tag{5}$$

(ii) Fully implicit(FI)

$$\frac{u^{n+1} - u^n}{\Delta t} = [\nu u_{xx} - u_x]^{n+1}, \tag{6}$$

(iii) Crank–Nicolson, half-step (CNHS)

$$\frac{u^* - u^n}{\Delta t} = \tfrac{1}{2}[-u_x]^* + \tfrac{1}{2}[-u_x]^n, \qquad u^*(0) = 1, \tag{7a}$$

$$\frac{u^{n+1} - u^*}{\Delta t} = \tfrac{1}{2}[\nu u_{xx}]^{n+1} + \tfrac{1}{2}[\nu u_{xx}]^*, \qquad \begin{aligned} u^{n+1}(0) &= 1, \\ u^{n+1}(1) &= 0 \text{ (P1)}, \\ u_x^{n+1}(1) &= 0 \text{ (P2)}, \end{aligned} \tag{7b}$$

and

(iv) Fully implicit, half-step (FIHS)

$$\frac{u^* - u^n}{\Delta t} = [-u_x]^*, \qquad u^*(0) = 1, \tag{8a}$$

$$\frac{u^{n+1} - u^*}{\Delta t} = [\nu u_{xx}]^{n+1}, \qquad \begin{aligned} u^{n+1}(0) &= 1, \\ u^{n+1}(1) &= 0 \text{ (P1)}, \\ u_x^{n+1}(1) &= 0 \text{ (P2)}, \end{aligned} \tag{8b}$$

where superscripts n, $n+1$, etc., denote the time level. In the CNHS and FIHS fractional step methods, the outflow boundary condition is imposed only on the second half-step in which the diffusive effects are invoked. The significance of this fact will become clear shortly. These time-differencing schemes—being fully or semi-implicit—are not subject to the Courant restriction on the magnitude of Δt. Schemes FI and FIHS are, however, only first order in time.

With fixed spatial resolution, $N = 64$, steady solutions (3a) and (3b) have been sought for $\nu = 30.0 \times 10^{-3}$, 0.6×10^{-3}, and 0.1×10^{-3}, corresponding to succeedingly less well-resolved boundary layers of thickness $O(\nu)$.[1] Tables I and II summarize the results for problems 1 and 2, respectively. The accuracy of the spectral solution is expressed in terms of relative errors between the computed steady field and the analytic result on the collocation grid; both maximum pointwise and rms errors are tabulated. For comparison, problems 1 and 2 have been solved using second-order centered finite differences ($\Delta x = 0.01$) and the steady-state errors also listed in these tables.

For well-resolved outflow layers (P1, $\nu = 30.0 \times 10^{-3}$), both spatial approximations are quite accurate, particularly in combination with the single-step schemes (5) and (6), both of which tend to give identical steady-state solutions. In addition, errors generally decrease with decreasing time step. For marginally resolved layers ($\nu = 0.6 \times 10^{-3}$), rather large rms errors (several tens of percent) are associated with the CN and FN methods. Not only do the fractional step schemes do somewhat better, but the error norms now typically increase for decreasing time step. The maximum pointwise error generally occurs near $x = 1$. Nevertheless, errors are substantial in the interior, as the rms figures indicate. Finally, for sufficiently small ν, the spectral integrations fail. Note, however, that the half-step schemes have a wider range of stability. For $\nu = 1.0 \times 10^{-4}$, FIHS differencing gives quite good rms results, considering the thinness of the boundary layer, although it does so only for the larger of the two time steps studied. The reason for this Δt stability dependence will be demonstrated shortly. For comparably small ν, the finite-difference calculations are stable but inaccurate.

[1] The central value, $\nu = 0.6 \times 10^{-3}$, has been chosen so that the e-folding length of the solution is equal to the shortest grid spacing on the Chebyshev collocation grid.

TABLE I[a]

(a) Spectral model

Time stepping	ν	($\Delta t = 2.0 \times 10^{-4}$) Maximum pointwise error	rms error	($\Delta t = 2.0 \times 10^{-3}$) Maximum pointwise error	rms error
CN	30.0×10^{-3}	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$
I		$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$
CNHS		$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$
FIHS		6.5×10^{-3}	1.9×10^{-3}	7.1×10^{-2}	2.1×10^{-2}
CN	0.6×10^{-3}	4.5×10^{-1}	2.7×10^{-1}	4.5×10^{-1}	2.7×10^{-1}
FI		4.5×10^{-1}	2.7×10^{-1}	4.5×10^{-1}	2.7×10^{-1}
CNHS		3.9×10^{-1}	2.3×10^{-1}	1.4×10^{-1}	8.4×10^{-2}
FIHS		1.6×10^{-1}	9.6×10^{-2}	4.9×10^{-1}	6.6×10^{-2}
CN	0.1×10^{-3}	unbounded		unbounded	
FI		unbounded		unbounded	
CNHS		unbounded		unbounded	
FIHS		unbounded		3.1×10^{-1}	1.1×10^{-1}

(b) Finite-difference model

Time Stepping	ν	($\Delta t = 2.0 \times 10^{-3}$) Maximum pointwise error	rms error	($\Delta t = 2.0 \times 10^{-2}$) Maximum pointwise error	rms error
CN	30.0×10^{-3}	8.1×10^{-3}	1.4×10^{-3}	8.1×10^{-3}	1.4×10^{-3}
FI		8.1×10^{-3}	1.4×10^{-3}	8.1×10^{-3}	1.4×10^{-3}
CNHS		9.1×10^{-2}	1.1×10^{-2}	8.5×10^{-1}	9.5×10^{-2}
FIHS		2.5×10^{-2}	5.7×10^{-3}	2.4×10^{-1}	5.1×10^{-2}
CN	0.6×10^{-3}	7.9×10^{-1}	1.3×10^{-1}	7.9×10^{-1}	1.3×10^{-1}
FI		7.9×10^{-1}	1.3×10^{-1}	7.9×10^{-1}	1.3×10^{-1}
CNHS		7.7×10^{-1}	1.3×10^{-1}	6.0×10^{-1}	9.7×10^{-2}
FIHS		7.6×10^{-1}	1.3×10^{-1}	5.2×10^{-1}	1.1×10^{-1}
CN	0.1×10^{-3}	9.3×10^{-1}	3.4×10^{-1}	9.3×10^{-1}	3.4×10^{-1}
FI		9.3×10^{-1}	3.4×10^{-1}	9.3×10^{-1}	3.4×10^{-1}
CNHS		9.2×10^{-1}	3.4×10^{-1}	8.9×10^{-1}	3.3×10^{-1}
FIHS		9.2×10^{-1}	3.4×10^{-1}	8.6×10^{-1}	3.3×10^{-1}

[a] Steady-state errors in the solution to problem 1 as a function of boundary layer thickness ν and time step Δt.

TABLE II[a]

Time stepping	ν	Spectral model ($\Delta t = 2.0 \times 10^{-3}$) Maximum pointwise error	Spectral model rms error	Finite-difference model ($\Delta t = 2.0 \times 10^{-2}$) Maximum pointwise error	Finite-difference model rms error
CN	0.6×10^{-3}	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$
FI		$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$
CNHS		$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$
FIHS		$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$
CN	0.1×10^{-3}	unbounded		1.6×10^{-3}	3.5×10^{-4}
FI		$< 10^{-6}$	$< 10^{-6}$	1.2×10^{-4}	6.6×10^{-5}
CNHS		$< 10^{-6}$	$< 10^{-6}$	3.1×10^{-4}	9.0×10^{-5}
FIHS		$< 10^{-6}$	$< 10^{-6}$	1.5×10^{-4}	8.8×10^{-5}
CN	0.01×10^{-3}	unbounded		4.5×10^{-1}	1.5×10^{-1}
FI		$< 10^{-6}$	$< 10^{-6}$	4.3×10^{-1}	1.5×10^{-1}
CNHS		1.8×10^{-6}	$< 10^{-6}$	4.5×10^{-1}	1.5×10^{-1}
FIHS		$< 10^{-6}$	$< 10^{-6}$	4.3×10^{-1}	1.6×10^{-1}
CN	0.001×10^{-3}	unbounded		9.2×10^{-1}	2.0×10^{-1}
FI		$< 10^{-6}$	$< 10^{-6}$	9.2×10^{-1}	2.6×10^{-1}
CNHS		8.5×10^{-6}	2.9×10^{-6}	9.2×10^{-1}	2.0×10^{-1}
FIHS		$< 10^{-6}$	$< 10^{-6}$	9.2×10^{-1}	2.6×10^{-1}

[a] Steady-state errors in the solution to problem 2 as a function of boundary layer thickness ν for the spectral and finite-difference models.

In general both the accuracy and range of stability of the spectral and finite-difference models are increased when derivative boundary conditions are specified on the outflow boundary (P2, Table II). The only exception occurs for the spectral model with CN differencing. This combination is still unstable for $\nu \leqslant 1.0 \times 10^{-4}$ whereas highly accurate solutions are obtained with the remaining differencing schemes for ν as small as 10^{-6}. As in P1, the finite-difference model can be quite inaccurate, despite the fact that the steady-state solution has no explicit boundary layer character.

4. Computational Stability of the Chebyshev Method

It is possible to explain the observed stability properties of the spectral computations by appealing to the following straightforward stability analysis. Let

$$u^{n+1} = Au^n, \tag{9}$$

$$\mathscr{L}_1\tilde{u}_{1j} = \left(-\frac{\partial}{\partial x}\right)\tilde{u}_{1j} = \lambda_{1j}\tilde{u}_{1j}, \qquad u(0) = 1, \tag{10a}$$

$$\mathscr{L}_2\tilde{u}_{2j} = \left(\nu\frac{\partial^2}{\partial x^2}\right)\tilde{u}_{2j} = \lambda_{2j}\tilde{u}_{2j}, \qquad \begin{aligned} u(0) &= 1, \\ u(1) &= 0 \text{ (P1)}, \\ u_x(1) &= 0 \text{ (P2)}, \end{aligned} \tag{10b}$$

and

$$\mathscr{L}_3\tilde{u}_{3j} = \left(\nu\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)\tilde{u}_{3j} = \lambda_{3j}\tilde{u}_{3j}, \qquad \begin{aligned} u(0) &= 1, \\ u(1) &= 0 \text{ (P1)}, \\ u_x(1) &= 0 \text{ (P2)}, \end{aligned} \tag{10c}$$

where A is the amplification factor for the given time and space differencing scheme and the $\mathscr{L}_i$, λ_{ij}, and $\tilde{u}_{ij}$ ($i = 1, 3; j = 1, N$) denote the specified linear operators and their associated eigenvalues and eigenvectors, under the assumed set of boundary conditions. There are, of course, N such eigenvalue/eigenvector pairs for each discrete operator.

Of interest to us here is that, for a fixed value of ν, the complex eigenvalues of operator $\mathscr{L}_3$ can have positive real parts only for N less than some critical value, say N_c. That is, there exists a value of N $(= N_c)$ which separates a region of $N > N_c$ for which all real (λ_{3j}) are negative, from another in which $N < N_c$ and at least one real (λ_{3j}) is positive. Tables III and IV tabulate the associated values of ν and N_c for boundary conditions (2a) and (2b), respectively; note the quasi-constant nature of the product $(\nu N_c^2)^{-1}$.

TABLE III[a]

ν	N_c	$(\nu N_c^2)^{-1}$
0.5×10^{-2}	15	0.88
1.25×10^{-3}	35	0.65
0.5×10^{-3}	61	0.54
0.3×10^{-3}	81	0.51
0.2×10^{-3}	101	0.49

[a] Corresponding values of the boundary layer thickness ν and critical resolution parameter N_c for problem 1.

The implication of the critical resolution parameter N_c for the stability of the Chebyshev method can be seen by considering the Crank–Nicolson scheme (5).

Under the above definitions of A_i and λ_{ij}, it is easily shown that the complex amplification factor for the jth eigenvector, u_{3j}, satisfies

$$| A |^2 = \frac{(1 + \rho_3)^2 + \sigma_3^2}{(1 - \rho_3)^2 + \sigma_3^2}, \tag{11}$$

where

$$\frac{\Delta t \lambda_{3j}}{2} = \rho_3 + i\sigma_3 .$$

It is quite clear that stability of this discrete formulation—that is, $| A |^2 \leqslant 1$—requires $\mathrm{Re}(\Delta t \lambda_{3j}/2) = \rho_3 \leqslant 0$ (all j). In other words, a minimum resolution, given by N_c, is required to maintain stable calculations when using the CN Chebyshev integration scheme. The results of the numerical tests verify this effect (Table Ia); for $N = 64$, stable integrations can be made for $\nu = 6.0 \times 10^{-4}$, but not for $\nu = 1.0 \times 10^{-4}$. The critical nondimensional parameter is the grid Reynolds number $(\nu N_c^2)^{-1}$ which in the case of P1 (P2) must be less than about 0.5 (1.3) for stability.

TABLE IV[a]

	N_c	$(\nu N_c^2)^{-1}$
1.25×10^{-3}	21	1.82
0.5×10^{-3}	37	1.47
0.3×10^{-3}	49	1.39
0.2×10^{-3}	61	1.35
0.1×10^{-3}	89	1.27

[a] Corresponding values of the boundary layer thickness ν and critical resolution parameter N_c for problem 2.

Formally, it can be shown that the N-term Chebyshev approximation to the advective–diffusive problem (1) and (2a) is convergent and weakly stable in that the growth rate associated with this approximation is bounded [8]. For practical purposes, however, the growth rate, which is of $O(1/\nu)$ for the problem addressed here, is sufficiently large as to make useful calculations impossible.

Similar arguments show that the less accurate fully implicit scheme can be made stable for $N < N_c$. Under definitions (10), the associated amplification factor becomes

$$| A |^2 = [(1 - \rho_3)^2 + \sigma_3^2]^{-1}, \tag{12}$$

where $\Delta t \lambda_{3j} = \rho_3 + i\sigma_3$. For stability,

$$(1 - \rho_3)^2 + \sigma_3^2 \geqslant 1.$$

Once again, $\rho_3 \leqslant 0$ ensures stability; however, for $0 \leqslant \rho_3 \ll 1$, the FI scheme can also be made stable if

$$\rho_3 \leqslant \sigma_3^2/2,$$

or equivalently,

$$\Delta t \geqslant \frac{2\,\mathrm{Re}(\lambda_{3j})}{[\mathrm{Im}(\lambda_{3j})]^2} \qquad (\text{all } j). \tag{13}$$

Given a sufficiently large time step, there exists a range of positive ρ_3 for which this computational problem is stable. Note that condition (13) must be satisfied for all N eigenvalues corresponding to the discrete spectrum of operator $\mathscr{L}_3$. Therefore, the FI scheme can be stabilized but perhaps only by increasing Δt and, hence, making it less accurate.

By using the multistep time-differencing schemes one might hope to avoid or relax the grid Reynolds number restriction. In theory, this should be possible because the discrete operator $\mathscr{L}_1 = -\partial/\partial x$ is well behaved if boundary conditions are specified at inflow ($x = 0$) only. Consequently, each of the half-steps making up the FIHS and CNHS schemes are individually stable, and, by the rule of thumb for the stability of fractional step schemes, this should guarantee the stability of the overall multistep method. Unfortunately, this need not be the case. For instance, the CNHS integration of P1 is computationally unstable for $\nu = 1.0 \times 10^{-4}$ (Table Ia).

The origin of this temporal instability which develops for $N < N_c$ (even with fractional step differencing) is the outflow boundary layer at $x = 1$. For any values of ν and $u > 0$, fractional step (8a) is algebraically stable, as is (8b). Nevertheless, Gottlieb and Orszag [8] have shown that this does not necessarily guarantee the stability of the total step if the discrete forms of the advective and diffusive operators do not commute, as in this example. These fractional step schemes may, in some cases, be stabilized by using a sufficiently large time step. Nevertheleess, Chebyshev approximation to the advective–diffusive equation (1) generally yields multistep methods in which, although each fractional step is algebraically stable, the total step is unstable for N less than some critical value.

5. The Effects of Nonlinearity

We briefly show that analogous computational restrictions apply to the integration of *nonlinear* transport equations by the Chebyshev collocation method. Consider the nonlinear version of problem 1, that is,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \qquad (0 \leqslant x \leqslant 1) \tag{14}$$

with

$$u(0, t) = 1, \qquad u(1, t) = 0, \qquad \text{Problem 3 (P3)}. \tag{15}$$

The nonlinear steady-state solution

$$u(x, t) \xrightarrow[t\to\infty]{} \frac{\tanh[(1-x)/2 - \nu]}{\tanh[1/2 - \nu]} \tag{16}$$

once again exhibits a narrow viscous boundary layer at the downstream end.

To solve Burgers equation (14) subject to the inhomogeneous boundary conditions (15), it is convenient to define the following two modified Crank–Nicolson time-differencing schemes:

(i) Nonlinear Crank–Nicolson (NCN)

$$\frac{u^{n+1} - u^n}{\Delta t} = \tfrac{1}{2}[\nu u_{xx} - \hat{u} u_x]^{n+1} + \tfrac{1}{2}[\nu u_{xx} - \hat{u} u_x]^n + (u - \hat{u})^n u_x{}^n, \tag{17}$$

and

(ii) Nonlinear Crank–Nicolson, half-step (NCNHS),

$$\frac{u^* - u^n}{\Delta t} = -\tfrac{1}{2}[\hat{u} u_x]^* - \tfrac{1}{2}[\hat{u} u_x]^n + (u - \hat{u})^n u_x{}^n, \qquad u^*(0) = 1, \tag{18a}$$

$$\frac{u^{n+1} - u^*}{\Delta t} = \tfrac{1}{2}[\nu u_{xx}]^{n+1} + \tfrac{1}{2}[\nu u_{xx}]^*, \qquad \begin{array}{l} u^{n+1}(0) = 1, \\ u^{n+1}(1) = 0, \end{array} \tag{18b}$$

where the reference field

$$\hat{u}(x) = 1 - x. \tag{19}$$

As is clear from (17) and (18), the intent of the NCN and NCNHS methods is to split off and treat implicitly a known part of the advective term—i.e., $\hat{u} u_x$—which may be thought of as that part arising due to nonzero values of u at the boundaries of the computational domain. In so doing, we greatly relax the Courant limitation on the maximally allowable time step, which for these schemes is dictated by the effective signal speed in the vicinity of the highly resolved regions near $x = 0$ and 1. While the specific form adopted for $\hat{u}(x)$ is somewhat arbitrary, the reference velocity given by (19) is particularly advantageous because the operator $(1 - x)(\partial/\partial x)$ can be compactly expressed in the space of Chebyshev polynomials [1]. The NCN and NCNHS time-stepping schemes can therefore be implemented quite efficiently.

Using these modified Crank–Nicolson schemes, we have sought stable solutions to Burgers equation for several values of ν. That part of the advective term treated explicitly has been calculated using the transform method [9]; the resulting nonlinear products are therefore alias-free. In all integrations, the time step has been chosen on the basis of the Courant condition, the known collocation grid spacing, and the effective (explicitly treated) velocity distribution $(u - \hat{u})$. The results of these calculations are shown in Table V.

As expected, for well-resolved downstream boundary layers ($\nu = 30.0 \times 10^{-3}$)

accurate results are obtained throughout the computational domain. For marginally resolved layers ($\nu = 0.6 \times 10^{-3}$), solutions proved to be stable, but nonsteady. The relative accuracy of the NCN and NCNHS schemes therefore cannot be determined for this range of ν. As in the case of the linearized advective equation, the spectral calculations are invariably unstable for sufficiently narrow downstream boundary layers. Also consistent with the linearized results is the fact that single- and multistep

TABLE V

Time stepping	ν	Δt	Maximum positive error	rms error
NCN	30.0×10^{-3}	2.0×10^{-3}	$<10^{-6}$	$<10^{-6}$
NCNHS			1.7×10^{-2}	5.4×10^{-3}
NCN	0.6×10^{-3}	2.0×10^{-4}	4.6×10^{-1}	2.5×10^{-1}*
NCNHS			4.4×10^{-1}	2.4×10^{-1}*
NCN	0.1×10^{-3}	1.0×10^{-4}	unbounded	
NCNHS			unbounded	

[a] Steady-state errors in the solution to the nonlinear Problem 3 as a function of boundary layer thickness ν for the spectral model. An asterisk (*) denotes a stable, but nonsteady, solution; for these cases, the errors have been measured relative to the computed solution at $t = 5$.

semi-implicit time-differencing schemes appear to be equally subject to this instability. The grid Reynolds number at which instability occurs is approximately the same as that noted above for Eq. (1). Related calculations not reported here indicate that explicit treatment of the advective effects—corresponding to the choice $\hat{u}(x) = 0$—leads to minimal stabilization of the spectral calculations while necessitating a very small time step.

6. Conclusion

For viscous transport problems, the presence of narrow downstream boundary layers arising from the imposed boundary conditions will not in general contaminate the interior solution as long as the boundary layer itself is well resolved. As our examples demonstrate, poor resolution of these outflow layers will result in inaccurate interior solutions for a wide range of spatial and temporal differencing schemes. More importantly, the Chebyshev collocation method we have examined is always unstable for sufficiently thin viscous layers. Neither the effects of nonlinearity nor the use of fractional step time-differencing methods removes this instability.

Inaccuracies in the interior solution due to poor outflow boundary layer resolution can be minimized in several ways. First, the physical problem can be locally modified to thicken the downstream boundary layer, perhaps by using a spatially dependent diffusive parameter, $\nu = \nu(x)$. Second, the discrete (numerical) problem can be optimized: (1) by specifying derivative rather than homogeneous boundary conditions on outflow, or (2) by adopting fully implicit or fractional step time-differencing methods. In this latter regard, it is interesting to note that the least accurate time-differencing schemes (in the formal sense) can yield the most accurate results for poorly resolved outflow boundary layers and that stability of these schemes may sometimes require a larger, rather than a smaller, time step. And although the fractional step methods are generally the most stable and accurate for small ν, their stability cannot be guaranteed (at least for the Chebyshev model) by the stability of their component steps.

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